Matrix Algebra – A Minimal Introduction

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Definition of a Linear Combination

Definition

- Suppose you have two predictors, x_1 and x_2
- The variable $\hat{y} = b_1 x_1 + b_2 x_2$ is said to be a *linear combination* of x_1 and x_2
- b_1 and b_2 are the *linear weights* which, in a sense, define a particular linear combination

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Linear Combinations in Regression

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- As we just saw, a linear model for y is a linear combination of one or more predictor variables, plus an intercept and an error term
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- Matrix algebra provides mathematical tools and notation for discussing linear models compactly

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Definition of A Matrix

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- A *matrix* is defined as an ordered array of numbers, of dimensions p, q.

Example

Below is a matrix $\mathbf{A}$ of dimensions 3×3.

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 4 \\ 7 & 1 & 8 \\ 6 & 6 & 0 \end{pmatrix}$$

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Matrix Notation

Notation

- My standard notation for a matrix $\mathbf{A}$ of order p, q will be:

$$ {}_p\mathbf{A}_q $$

- Note that in my notation, matrices and vectors are in boldface
- Gelman and Hill, presents matrices and vectors in math-italics, but their meaning will usually be clear from context

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Elements of a Matrix

Matrix Elements

- The individual numbers in a matrix are its *elements*
- We use the following notation to indicate that "$\mathbf{A}$ is a matrix with elements a_{ij} in the i, jth position"

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Matrix Notation
Subscript Notation

Subscript Notation

- When we refer to element a_{ij}, the *first* subscript will refer to the *row position* of the elements in the array
- The *second* subscript (regardless of which letter is used in this position) will refer to the column position.
- Hence, a typical matrix ${}_p\mathbf{A}_q$ will be of the form:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1q} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2q} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3q} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{p1} & a_{p2} & a_{p3} & \cdots & a_{pq} \end{pmatrix}$$

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Types of Matrices

On several subsequent slides, we will define a number of types of matrices that are referred to frequently in practice.

Types of Matrices
Rectangular Matrix

Rectangular Matrix

For any ${}_p\mathbf{A}_q$, if $p \neq q$, $\mathbf{A}$ is a *rectangular* matrix

Example

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}$$

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Square Matrix

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Types of Matrices

Lower Triangular Matrix

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For any square matrix $\mathbf{A}$, $\mathbf{A}$ is *lower triangular* if $a_{ij} = 0$ for $i < j$

Example

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 10 \end{pmatrix}$$

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Types of Matrices
Upper Triangular Matrix

Upper Triangular Matrix

For any square matrix $\mathbf{A}$, $\mathbf{A}$ is *upper triangular* if $a_{ij} = 0$ for $i > j$

Example

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{pmatrix}$$

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Types of Matrices

Diagonal Matrix

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For any square matrix $\mathbf{A}$, $\mathbf{A}$ is a *diagonal matrix* if $a_{ij} = 0$ for $i \neq j$

Example

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Types of Matrices
Scalar Matrix

Scalar Matrix
For any diagonal matrix **A**, if all diagonal elements are equal, **A** is a *scalar* matrix

Example
$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

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Types of Matrices

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For any scalar matrix **A**, if all diagonal elements are 1, **A** is an *identity* matrix

Example

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Types of Matrices
Symmetric Matrix

Square Matrix

A square matrix $\mathbf{A}$ is *symmetric* if $a_{ji} = a_{ij} \ \forall \ i, j$

Example

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

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A square matrix $\mathbf{A}$ is *symmetric* if $a_{ji} = a_{ij} \ \forall \ i, j$

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Null Matrix

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Row Vector

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- A *row vector* is a matrix with only one row
- It is common to identify row vectors in matrix notation with lower-case boldface and a "prime" symbol, like this

$$\mathbf{a}'$$

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Matrix Operations

Matrix Operations

- Two matrix operations, addition and subtraction, are essentially the same as their familiar scalar equivalents
- But multiplication and division are rather different!
  - There is only a limited notion of division in matrix algebra, and
  - Matrix multiplication shares some properties with scalar multiplication, but in other ways is dramatically different
- We will try to keep reminding you where you need to be careful!

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Matrix Addition

Matrix Addition

- For two matrices $\mathbf{A}$ and $\mathbf{B}$ to be *conformable* for addition or subtraction, they must have the same numbers of rows and columns
- To add two matrices, simply add the corresponding elements together

Example

$$\begin{pmatrix} 1 & 4 & 1 \\ 1 & 3 & 3 \\ 3 & 0 & 5 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \\ 3 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 6 & 2 \\ 3 & 5 & 5 \\ 6 & 2 & 5 \end{pmatrix}$$

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Properties of Matrix Addition

Matrix addition has some important mathematical properties, which, fortunately, mimic those of scalar addition and subtraction. Consequently, there is little "negative transfer" involved in generalizing from the scalar to the matrix operations.

Properties of Matrix Addition

For matrices **A**, **B**, and **C**, properties include:

- *Associativity.* $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$
- *Commutativity.* $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$

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Scalar Multiple

Scalar Multiple

- When we multiply a matrix by a scalar, we are computing a *scalar multiple*, not to be confused with a scalar product, which we will learn about subsequently
- To compute a scalar multiple, simply multiply every element of the matrix by the scalar

Example

$$2\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 4 \\ 4 & 2 \end{pmatrix}$$

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Properties of Scalar Multiplication

For matrices $\mathbf{A}$ and $\mathbf{B}$, and scalars a and b, scalar multiplication has the following mathematical properties:

Properties of Scalar Multiplication

- $(a+b)\mathbf{A} = a\mathbf{A} + b\mathbf{A}$
- $a(\mathbf{A}+\mathbf{B}) = a\mathbf{A} + a\mathbf{B}$
- $a(b\mathbf{A}) = (ab)\mathbf{A}$
- $a\mathbf{A} = \mathbf{A}a$

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Scalar Product

Scalar Product

- Given a row vector $\mathbf{x}'$ and a column vector $\mathbf{y}$ having q elements each
- The *scalar product* $\mathbf{x}'\mathbf{y}$ is a scalar equal to the sum of cross-products of the elements of $\mathbf{x}'$ and $\mathbf{y}$.

Example

If $\mathbf{x}' = \begin{pmatrix} 1 & 2 & 1 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$ then

$\mathbf{x}'\mathbf{y} = (1)(2) + (2)(3) + (1)(2) = 10$

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If $\mathbf{x}' = \begin{pmatrix} 1 & 2 & 1 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$ then

$\mathbf{x}'\mathbf{y} = (1)(2) + (2)(3) + (1)(2) = 10$

Scalar Product

Scalar Product

- Given a row vector $\mathbf{x}'$ and a column vector $\mathbf{y}$ having q elements each
- The *scalar product* $\mathbf{x}'\mathbf{y}$ is a scalar equal to the sum of cross-products of the elements of $\mathbf{x}'$ and $\mathbf{y}$.

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Matrix Transposition

Transposing a matrix is an operation which plays a very important role in multivariate statistical theory. The operation, in essence, switches the rows and columns of a matrix.

Matrix Transposition

Let ${}_p\mathbf{A}_q = \{a_{ij}\}$. Then the *transpose* of $\mathbf{A}$, denoted $\mathbf{A}'$, is defined as

$${}_q\mathbf{A}'_p = \{a_{ji}\}$$

Example

If ${}_2\mathbf{A}_3 = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$, then ${}_3\mathbf{A}'_2 = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$

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Matrix Transposition

Some Key Properties

Key Properties of Matrix Transposition

- $(\mathbf{A}')' = \mathbf{A}$
- $(c\mathbf{A})' = c\mathbf{A}'$
- $(\mathbf{A} + \mathbf{B})' = \mathbf{A}' + \mathbf{B}'$
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- A square matrix $\mathbf{A}$ is symmetric if and only if $\mathbf{A} = \mathbf{A}'$

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Matrix Multiplication
Conformability

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- *order matters* in matrix multiplication.

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- That is, the matrix product $\mathbf{AB}$ need not be the same as the matrix product $\mathbf{BA}$.
- Indeed, the matrix product $\mathbf{AB}$ might be well-defined, while the product $\mathbf{BA}$ might not exist.
- When we compute the product $\mathbf{AB}$, we say that $\mathbf{A}$ is *post-multiplied* by $\mathbf{B}$, or that $\mathbf{B}$ is *premultiplied* by $\mathbf{A}$

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Matrix Multiplication

Dimension of a Matrix Product

If two or more matrices are conformable, there is a strict rule for determining the dimension of their product

Matrix Multiplication — Dimensions of a Product

- The product ${}_p\mathbf{A}_q\mathbf{B}_r$ will be of dimension $p \times r$
- More generally, the product of any number of conformable matrices will have the number of rows in the leftmost matrix, and the number of columns in the rightmost matrix.
- For example, the product ${}_p\mathbf{A}_q\mathbf{B}_r\mathbf{C}_s$ will be of dimensionality $p \times s$

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Three Approaches

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- Matrix multiplication might well be described as the key operation in matrix algebra
- What makes matrix multiplication particularly interesting is that there are numerous lenses through which it may be viewed
- We shall examine 3 ways of "looking at" matrix algebra
- All of them rely on *matrix partitioning*, which we'll examine briefly in the next 2 slides

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Matrix Multiplication
The Row by Column Approach

Partitioning a Matrix into Rows

- Any $p \times q$ matrix $\mathbf{A}$ may be partitioned into as a set of p rows
- For example, the 2×3 matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix}$$

  may be thought of as two rows, $(1 \quad 2 \quad 3)$ and $(3 \quad 3 \quad 3)$ stacked on top of each other
- We have a notation for this. We write

$$\mathbf{A} = \begin{pmatrix} \mathbf{a}_1' \\ \mathbf{a}_2' \end{pmatrix}$$

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where, for example,

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- Suppose you wish to multiply the two matrices $\mathbf{A}$ and $\mathbf{B}$, where
$$\mathbf{A} = \begin{pmatrix} 2 & 7 \\ 3 & 5 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}$$
- You know that the product, $\mathbf{C} = \mathbf{AB}$, will be a 2×3 matrix
- Partition $\mathbf{A}$ into 2 rows, and $\mathbf{B}$ into 3 columns.
- Element $c_{i,j}$ is the scalar product of row i of $\mathbf{A}$ with column j of $\mathbf{B}$

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Matrix Multiplication
The Row by Column Approach

Again suppose you wish to compute the product $\mathbf{C} = \mathbf{AB}$ using the matrices from the preceding slide.

Example

Compute $c_{1,1}$.

$$\begin{pmatrix} 2 & 7 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}$$

Taking the product of the row 1 of $\mathbf{A}$ and column 1 of $\mathbf{B}$, we obtain $(2)(1) + (7)(2) = 16$

Matrix Multiplication
The Row by Column Approach

Again suppose you wish to compute the product $\mathbf{C} = \mathbf{AB}$ using the matrices from the preceding slide.

Example

Compute $c_{2,3}$.

$$\begin{pmatrix} 2 & 7 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}$$

Taking the product of the row 2 of $\mathbf{A}$ and column 3 of $\mathbf{B}$, we obtain $(3)(1) + (5)(3) = 18$

Matrix Multiplication

Linear Combination of Columns Approach

Linear Combination of Columns

- When you post-multiply a matrix $\mathbf{A}$ by a matrix $\mathbf{B}$, each column of $\mathbf{B}$ generates, in effect, a column of the product $\mathbf{AB}$
- Each column of $\mathbf{B}$ contains a set of linear weights
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Matrix Multiplication
Linear Combination of Columns Approach

Linear Combination of Columns

- Consider the product

$$\begin{pmatrix} 2 & 7 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}$$

- The first column of the product is produced by applying the linear weights 1 and 2 to the columns of the first matrix
- The result is

$$1 \begin{pmatrix} 2 \\ 3 \end{pmatrix} + 2 \begin{pmatrix} 7 \\ 5 \end{pmatrix} = \begin{pmatrix} 16 \\ 13 \end{pmatrix}$$

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$$\begin{pmatrix} 2 & 7 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} {\color{red}1} & 2 & 1 \\ {\color{red}2} & 2 & 3 \end{pmatrix}$$

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Matrix Multiplication
Linear Combination of Columns Approach

Linear Combination of Columns

- Consider once again the product

$$\begin{pmatrix} 2 & 7 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}$$

- The second column of the product is produced by applying the linear weights 2 and 2 to the columns of the first matrix
- The result is

$$2\begin{pmatrix} 2 \\ 3 \end{pmatrix} + 2\begin{pmatrix} 7 \\ 5 \end{pmatrix} = \begin{pmatrix} 18 \\ 16 \end{pmatrix}$$

Matrix Multiplication

Linear Combination of Columns Approach

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Matrix Multiplication
Mathematical Properties

The following are some key properties of matrix multiplication:

Mathematical Properties of Matrix Multiplication

- *Associativity.*
$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$$
- *Not generally commutative.* That is, often $\mathbf{AB} \neq \mathbf{BA}$.
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$$\mathbf{C}(\mathbf{A}+\mathbf{B}) = \mathbf{CA} + \mathbf{CB}$$
- Assuming it is conformable, the identity matrix $\mathbf{I}$ functions like the number 1, that is ${}_p\mathbf{A}_q\mathbf{I}_q = \mathbf{A}$, and ${}_p\mathbf{I}_p\mathbf{A}_q = \mathbf{A}$.
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Inverse of a Square Matrix

Definition

Matrix Inverse

- A $p \times p$ matrix has an inverse if and only if it is square and of full rank, i.e., i.e., no column of A is a linear combination of the others.
- If a square matrix $\mathbf{A}$ has an inverse, it is the unique square matrix $\mathbf{A}^{-1}$ such that

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- $(\mathbf{A}')^{-1} = (\mathbf{A}^{-1})'$
- If $\mathbf{A} = \mathbf{A}'$, then $\mathbf{A}^{-1} = (\mathbf{A}^{-1})'$
- The inverse of the product of several invertible square matrices is the product of their inverses in reverse order. For example
$$(\mathbf{ABC})^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$$
- For nonzero scalar c, $(c\mathbf{A})^{-1} = (1/c)\mathbf{A}^{-1}$
- For diagonal matrix $\mathbf{D}$, $\mathbf{D}^{-1}$ is a diagonal matrix with diagonal elements equal to the reciprocal of the corresponding diagonal elements of $\mathbf{D}$.

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Uses

Systems of Linear Equations

- Suppose you have the two simultaneous equations $2x_1 + x_2 = 5$, and $x_1 + x_2 = 3$.
- These two equations may be expressed in matrix algebra in the form

$$\mathbf{Ax} = \mathbf{b},$$

  or

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

- We wish to solve for x_1 and x_2, which amounts to solving $\mathbf{Ax} = \mathbf{b}$ for $\mathbf{x}$

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Systems of Linear Equations

Example

In the previous numerical example, we had

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \text{ and } \mathbf{b} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

It is easy to see that

$$\mathbf{A}^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

and so

$$\mathbf{x} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Trace of a Matrix

Trace of a Matrix

- The *trace* of a square matrix $\mathbf{A}$, $Tr(\mathbf{A}$, is the sum of its diagonal elements
- The trace is often employed in matrix algebra to compute the sum of squares of all the elements of a matrix
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$$Tr(\mathbf{A}\mathbf{A}') = \sum_i \sum_j A_{i,j}^2$$

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